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The Structure of Projection-Valued States: A Generalization of Wigner's Theorem[1]

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Abstract

A projection-valued state is defined to be a completely orthoadditive map from the projections on one Hilbert space into the projections on another Hilbert space, which preserves the unit. Any such mapping is shown to have the form $P \to U_1(P \otimes 1_1)U_1^{-1} \oplus U_2(P \otimes 1_2)U_2^{-1}$, where U_1 is unitary and U_2 is antiunitary, generalizing Wigner's theorem on symmetry transformations. A physical interpretation is given and the relation to "quantum logic" is discussed.

Let $\mathscr{H}$ and $\mathscr{H}'$ be complex Hilbert spaces with dim $\mathscr{H} > 2$ and dim $\mathscr{H}' > 0$; let $\mathbb{P}(\mathscr{H})$ and $\mathbb{P}(\mathscr{H}')$ be the projection lattices and let $\mathbb{P}_0(\mathscr{H})$ and $\mathbb{P}_0(\mathscr{H}')$ be the atomic projections. Wigner's theorem (see Lomont and Mendelson, 1963; Uhlhorn, 1963; Emch and Piron, 1963; Bargmann, 1964) can be stated as follows:

> *Theorem 1 (Wigner's Theorem).* If ϕ maps $\mathbb{P}_0(\mathscr{H})$ onto $\mathbb{P}_0(\mathscr{H}')$ such that for all P and Q in $\mathbb{P}_0(\mathscr{H})$, $\| \phi(P)\phi(Q) \| = \| PQ \|$, then there exists a mapping $U: \mathscr{H} \to \mathscr{H}'$ such that U is either unitary or antiunitary and $\phi(P) = UPU^{-1}$.

In analogy with a (numerical) state on $\mathbb{P}(\mathscr{H})$, which is a completely orthoadditive map $\rho : \mathbb{P}(\mathscr{H}) \to [0, 1]$ such that $\rho(1) = 1$, let us define a *projection-valued state* (or *PV state*) on $\mathbb{P}(\mathscr{H})$ into $\mathbb{P}(\mathscr{H}')$ to be a completely orthoadditive map $\phi : \mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H}')$ such that $\phi(1) = 1$. [It is more customary to require states to be only countably orthoadditive, but it has been shown by Eilers and Horst (1975) that if dim $\mathscr{H}$ is not an inaccessible cardinal (in particular for the cases $\aleph_0$, c and 2^c), then a countably orthoadditive state is automatically completely orthoadditive. By noting that the composition of a state with a PV state is again a state, it is easy to show that

[1] The contents of this paper are a portion of the author's dissertation at the University of Massachusetts at Amherst.

the result also obtains for PV states.] By Corollary 1 below, Wigner's theorem can then be restated as follows:

Theorem 1a (Wigner's Theorem). If ϕ is a PV state on $\mathbb{P}(\mathscr{H})$ into $\mathbb{P}(\mathscr{H}')$ which preserves atoms, then there exists a mapping U: $\mathscr{H} \to \mathscr{H}'$ such that U is either unitary or antiunitary and $\phi(P) = UPU^{-1}$.

In this form Wigner's theorem is an immediate consequence (via a dimension argument) of the following structure theorem for PV states.

Theorem 2. Let ϕ be a PV state on $\mathbb{P}(\mathscr{H})$ into $\mathbb{P}(\mathscr{H}')$. Then ϕ can be uniquely extended to a *-linear Jordan isometry $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}')$. Furthermore, there exist mutually orthogonal subspaces $\mathscr{H}'_1$ and $\mathscr{H}'_2$ of $\mathscr{H}'$ with $\mathscr{H}' = \mathscr{H}'_1 \oplus \mathscr{H}'_2$, there exist Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ (possibly of dimension 0), and there exists a unitary map U_1: $\mathscr{H} \otimes \mathscr{H}_1 \to \mathscr{H}_1$ and an antiunitary map U_2: $\mathscr{H} \otimes \mathscr{H}_2 \to \mathscr{H}'_2$ such that

$$\Phi(T) = U_1(T \otimes 1_1)U_1^{-1} + U_2(T^* \otimes 1_2)U_2^{-1}$$

where 1_i is the unit on $\mathscr{H}_i$.

The proof of the theorem will be given using the following two lemmas.

Lemma 1. ϕ can be uniquely extended to a *-linear, norm-continuous, Jordan map $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}')$.

Proof. For $A = A^*$, let $A = \int_\Lambda \lambda dE_\lambda$ and define $\Phi(A) = \int_\Lambda \lambda d\phi \circ E_\lambda$; for $T \neq T^*$ put $\Phi(T) = \Phi((T + T^*)/2) + i\Phi((T - T^*)/2i)$. For each density operator (positive operator with trace 1) D' on $\mathscr{H}'$, there exists a density operator D on $\mathscr{H}$ such that for all T in $\mathscr{B}(\mathscr{H})$, $\mathrm{Tr}(TD) = \mathrm{Tr}(\Phi(T)D')$: For projections this follows from Gleason's theorem (see Eilers and Horst, 1975; also Gleason, 1957), since for every state ρ' on $\mathbb{P}(\mathscr{H}')$ $\rho' \circ \phi$ is a state on $\mathbb{P}(\mathscr{H})$; for self-adjoint operators use $\mathrm{Tr}[(\int_\Lambda \lambda dE_\lambda)D] = \int_\Lambda \lambda d[\mathrm{Tr}(E_\lambda D)]$ (see Jauch, 1968, p. 132). Thus for every density D' on $\mathscr{H}'$, $\mathrm{Tr}[D'(\Phi(\tau_1 T_1 + \tau_2 T_2))] = \mathrm{Tr}[D'(\tau_1 \Phi(T_1) + \tau_2 \Phi(T_2))]$, whence Φ is linear (use Corollary on p. 296 of Rudin, 1973).

Now $A = A^*$ implies $\Phi(A) = \Phi(A)^*$ (Halmos, 1957, Theorem 2, p. 60), so by linearity $\Phi(T) = \Phi(T)^*$. Also, $A = A^*$ implies $\|\Phi(A)\| =$ (spectral radius $\Phi(A)$) $\leqslant$ (spectral radius A) $= \|A\|$ and thus $\|\Phi(T)\| \leqslant 2\|T\|$, so Φ is norm continuous. Uniqueness then follows since the spectral integral is norm convergent.

It remains to show that Φ is a Jordan map (i.e., preserves squares). For self-adjoint operators this follows from $(\int \lambda dE_\lambda)^2 = \int \lambda^2 dE_\lambda$ and the norm-convergence of the spectral integral. For general T, consider $A_\theta = e^{i\theta}T + e^{-i\theta}T^*$ and use $\Phi(A_\theta{}^2) = \Phi(A_\theta)^2$ and linearity to obtain an identity; substitute $\theta = 0, \pi/4, \pi/2$ and solve the resulting three equations to get $\Phi(T^2) = \Phi(T)^2$. ∎

Lemma 2. Let $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}')$ be a *-homomorphism (respectively, *-antihomomorphism) which is an extension of a PV state. Then there exists a Hilbert space $\mathscr{H}_1$ and a unitary (respectively, antiunitary) map $U: \mathscr{H} \otimes \mathscr{H}_1 \to \mathscr{H}'$ such that

$$\Phi(T) = U(T \otimes 1_1)U^{-1} \text{ [respectively, } \Phi(\tau) = U(T^* \otimes 1_1)U^{-1}\text{]},$$

where 1_1 is the unit on $\mathscr{H}_1$.

Proof. Assume Φ is a homomorphism and let $\{e_0, e_1, \ldots\}$ be an orthonormal basis for $\mathscr{H}$; let P_i be the projection onto e_i and let $\mathscr{H}_1 = \Phi(P_0)(\mathscr{H}')$. Show that $T_1 e_0 = T_2 e_0$ implies that for all y in $\mathscr{H}_1, \Phi(T_1)y = \Phi(T_2)y$. Define $U(e_i, f_j) = \Phi(T_i)f_j$, where T_i is any element of $\mathscr{B}(\mathscr{H})$ with $e_i = T_i e_0$ and $\{f_j\}$ is an orthonormal basis for $\mathscr{H}_1$. Show that $\{U(e_i, f_j)\}$ is an orthonormal set: Use the fact that T_i can be chosen to be unitary and that $T_i e_0 = e_i$ implies that $P_i T_i e_0 = e_i$. Show that $\{U(e_i, f_j)\}$ is an orthonormal basis: Let $Q_{i,j}$ be the projection onto $U(e_i, f_j)$; if Q_j is the projection onto f_j and $U(e_i, f_j) = \Phi(U_i)f_j$, where U_i is unitary, then $Q_{i,j} = \Phi(U_i)Q_j\Phi(U_i)^*$ and $1 = \Sigma\, Q_{i,j}$. Extend U to all of $\mathscr{H} \otimes \mathscr{H}_1$. Then $\Phi(T)U = U(T \otimes 1_1)$ since both are linear and norm continuous, and they agree on a basis.

If Φ is an antihomomorphism, precede it by an antihomomorphism of the form $T \to JT^*J$, where J is a conjugation (see Stone, 1932, pp. 357-359), and apply the above argument to the composition. ■

Proof of Theorem 2. By Theorem 1, p. 153 of Emch (1972), there exist mutually orthogonal subspaces $\mathscr{H}'_1$ and $\mathscr{H}'_2$ of $\mathscr{H}'$ with $\mathscr{H}' = \mathscr{H}'_1 \oplus \mathscr{H}'_2$ and there exists a *-homomorphism $\Phi_1: \mathscr{H} \to \mathscr{H}'_1$ and a *-antihomomorphism $\Phi_2: \mathscr{H} \to \mathscr{H}'_2$ such that $\Phi = \Phi_1 + \Phi_2$. The theorem then follows from Lemma 2. ■

We are now in a position to show the equivalence of Theorems 1 and 1a.

Corollary 1. Let $\phi_0: \mathbb{P}_0(\mathscr{H}) \to \mathbb{P}_0(\mathscr{H}')$. Then ϕ_0 can be extended to a PV state iff ϕ_0 is onto and for all P and Q in $\mathbb{P}_0(\mathscr{H})$

$$\|PQ\| = \|\phi_0(P)\phi_0(Q)\|$$

Proof. If ϕ_0 can be extended to a PV state the conditions follow immediately from the theorem. Conversely, suppose that ϕ_0 satisfies the conditions. Use Bessel's equality (applied to atomic projections) to show that if Q and P_i $(i \in I)$ are in $\mathbb{P}_0(\mathscr{H})$ with $Q \leqslant \Sigma\, P_i$, then $\phi_0(Q) \leqslant \Sigma\, \phi_0(P_i)$. (Show that $\|\phi_0(Q) - \Sigma\phi_0(P_i)\phi_0(Q)\|^2 = 0$.) Define $\pi(\Sigma P_i) = \Sigma\phi_0(P_i)$. ■

The theorem admits a perspicuous physical interpretation. A quantum-mechanical system is represented by a Hilbert space; the self-adjoint operators are the observables of the system, and, in particular, the projections are yes-no questions or experimental propositions about the system (Birkhoff and von Neumann, 1936; Jauch, 1968; Varadarajan, 1968; Mackey, 1963). Thus a family (P_i) of (nonzero) mutually orthogonal projections with $1 = \Sigma P_i$ represents a maximal set of (nonantitautological) mutually exclusive propositions; such a family is called an experiment (cf. Kolmogorov, 1956, p. 6). A mapping $\phi: \mathbb{P}(\mathscr{H}) \to \mathbb{P}(\mathscr{H}')$ assigns to each proposition of the first system a proposition of the second system; ϕ is a PV state exactly when this assignment takes experiments on the first system over into experiments on the second system. Under these circumstances it is not surprising that ϕ can be extended to map observables on the first system into observables on the second system. The theorem then tells us that an experiment-preserving map must have a very special form. To see what this means physically, apply each of the mappings Φ_1 and Φ_2, separately, to a given observable T. The decomposition $\Phi_1(T) = U_1(T \otimes 1_1)U_1^{-1}$ says $\mathscr{H}'_1$ is equivalent to a compound physical system made up of the original system (represented by $\mathscr{H}$) and another system (represented by $\mathscr{H}_1$) (see Jauch, 1968, §§11.7 and 11.8). Similarly, $\Phi_2(T) = U_2(T \otimes 1_2)U_2^{-1}$ tells us that $\mathscr{H}'_2$ is equivalent to a compound system made up of the original system and a system represented by $\mathscr{H}_2$, but with an inversion in time (see Jauch, 1968, pp. 244 and 260; also Uhlhorn, 1963, §9). The mathematics thus allows us to make time reversals, but "warns" us perhaps by the orthogonality of $\mathscr{H}'_1$ and $\mathscr{H}'_2$ not to superpose states from the time-forward and time-backward parts of the system.

Thus, to stay in the realm of forward time, Φ_2 must be zero. There are other reasons to believe that Φ_2 will be zero in real physical situations. For example, while the product of the observables A and B does not in general have a direct physical interpretation, their Hermitian commutator $i[A, B] = i(AB - BA)$ is another observable—one of particular importance in quantum mechanics. If we insist that Φ preserve this commutator, this forces Φ_2 to be zero. In fact, if $H = H^*$ is the Hamiltonian for the system represented by $\mathscr{H}$, then requiring for each observable A that $i[H, A]$—which is interpreted as dA/dt (see Dirac, 1958, equation (13), p. 112)—be preserved still forces Φ_2 to be zero (unless $H = \lambda I$).

We can also look at the theorem from the opposite point of view. Suppose we are given two physical systems represented by $\mathscr{H}_1$ and $\mathscr{H}_2$; what Hilbert space $\mathscr{H}$ should represent the combined system? Normally physicists require two things (see Jauch, 1968, p. 175): (1) Every observable A_i on the component system $\mathscr{H}_i$ corresponds to an observable $\mathbb{A}_i$ on $\mathscr{H}$ in such a way that $\mathbb{A}_1$ always commutes with $\mathbb{A}_2$; and (2) the only observables on $\mathscr{H}$ that commute with all the $\mathbb{A}_1$'s and $\mathbb{A}_2$'s are ones of the form $\lambda 1$. (Neither of these assumptions is ever really explained experimentally.) If the dynamics are to be preserved, the theorem implies that $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, $\mathbb{A}_1 = A_1 \otimes 1_2$ and $\mathbb{A}_2 = 1_1 \otimes A_2$, that is, just what physicists always use.

Although the main physical interest is in the complex Hilbert spaces studied above, an analogous, but slightly stronger, result can be obtained in the real case: Every PV state on $\mathbb{P}(\mathscr{H})$ into $\mathbb{P}(\mathscr{H}')$ can be extended to a unique homomorphism of $\mathscr{B}(\mathscr{H})$ into $\mathscr{B}(\mathscr{H}')$ such that $T \to U(T \otimes 1)U^{-1}$, where U is unitary (use Theorem 1 of Martindale, 1967, in the place of Theorem 1 of Emch, 1972). Again the real version of Wigner's theorem is an immediate consequence. (See Uhlhorn, 1963, or Lomont and Mendelson, 1963, for statement and discussion of Wigner's theorem in this case.)

Another interesting consequence of Theorem 2 is the following:

Corollary 2. A PV state is a complete lattice monomorphism.

Proof. Φ_i restricted to $\mathbb{P}(\mathscr{H})$ is a complete lattice homomorphism onto its range; the range is a von Neumann algebra (Topping, 1971, Theorem 14, p. 59) and hence (Berberian, 1972, Proposition 8, p. 23 and Proposition 9, p. 24) the lattice operations agree with those in $\mathbb{P}(\mathscr{H}'_i)$. ϕ is injective since Φ is isometric. ∎

The definition of a PV state on $\mathbb{P}(\mathscr{H})$ into $\mathbb{P}(\mathscr{H}')$ can be extended verbatim to a PV state on a quantum logic L into $\mathbb{P}(\mathscr{H}')$. [For present purposes a quantum logic will mean a complete orthomodular lattice; see Varadarajan (1968), p. 105 and Jauch (1968), Chapter 5; compare Jeffcott (1972). We will also understand orthoadditivity to entail preservation of orthogonality: $P \perp Q$ implies $\phi(P) \perp \phi(Q)$.]

The structure theorem above does not extend to these more general PV states. In fact, PV states on a quantum logic can be very unruly in comparison with PV states on Hilbert-space lattices. For example, a PV state ϕ on a Hilbert-space lattice is seen to have the following properties: (i) $\phi(P) \perp \phi(Q)$ implies $P \perp Q$; (ii) ϕ is injective; (iii) P commutes with Q if and only if $\phi(P)$ commutes with $\phi(Q)$; and (iv) ϕ is a complete lattice homomorphism. (Note that commutativity can be characterized purely in terms of the lattice structure: P commutes with Q if and only if the sublattice generated by $\{P, Q, P', Q'\}$ is a Boolean algebra.) A PV state on a quantum logic need have none of these properties.

In the first place it is possible for a quantum logic to be reducible, i.e., expressible as a direct product of other quantum logics (see Birkhoff, 1967, p. 8). Physically, reducibility corresponds to the existence of superselection rules. In the traditional Hilbert-space formulation of quantum mechanics superselection rules are generally taken to divide the Hilbert space into a direct sum of "coherent subspaces." The appropriate logic is then the direct product of the projection lattices of the coherent subspaces (see the appendix to Mackey, 1963; Varadarajan, 1968, VII.4; or Jauch, 1968, p. 109). If $\mathscr{L}$ is a direct product of Hilbert-space projection lattices, a structure theorem for PV states on $\mathscr{L}$ can be proved by applying Theorem 2 to each of the factors. Any PV state on such an $\mathscr{L}$ is then seen to satisfy (iv), although if any of the factors is mapped into zero, (i)–(iii) are not satisfied. Clearly, this is a general phenomenon: It is always possible to violate conditions (i)–(iii) on a reducible quantum logic by mapping one of the factors into zero.

But even if no factor is mapped into zero (as, for example, when the quantum logic is irreducible), PV states need not satisfy any of the properties (i)–(iv). In fact, since a dispersion-free state [values in (0, 1)] can be seen as a PV state that virtually always fails to have properties (i)–(iv), any quantum logic that admits dispersion-free states will have erratic PV states. Since it is the superpositions that interconnect the elements of the lattice (and, for example, make Gleason's theorem work), it would seem that perhaps the superposibility in Hilbert-space lattices is what lends the uniformity to the PV states. This is, however, not the case, since there exist irreducible, indecomposable (Sumner, 1973), complete, atomic, orthomodular lattices satisfying the Superposition Principle (Jauch, 1968, p. 106), but admitting a strong set of dispersion-free states. [A set of states is strong if $P \nleq Q$ implies that there exists a state ρ in the set such that $\rho(P) = 1 > \rho(Q)$.]

It would appear therefore that the regularity of the PV states might be related to the nonexistence of dispersion-free states. Jauch's proof that there are no dispersion-free states on $\mathbb{P}(\mathscr{H})$ [i.e., that $\mathbb{P}(\mathscr{H})$ admits no "hidden variables"] relies on the fact that all the states in Hilbert space have the Jauch–Piron property, that is, $\rho(P) = \rho(Q) = 1$ implies $\rho(P \wedge Q) = 1$ [or equivalently, $\rho(P) = \rho(Q) = 0$ implies $\rho(P \vee Q) = 0$]. With respect to Jauch–Piron states the lattice-theoretic infima (suprema) behave like classical conjunctions (disjunctions): If P and Q are both true (false), then so is $P \wedge Q$ ($P \vee Q$), where "true" means "have probability 1" and "false" means "have probability 0." In some sense then quantum logics whose states are Jauch–Piron, are of a very classical nature; it has, in fact, been shown by Rüttimann (1977, Theorem 4.3) that a finite quantum logic with a strong set of Jauch–Piron states must be a Boolean algebra.

We are thus left with the following questions: Under what conditions do the PV states on a quantum logic satisfy conditions (i)–(iv)? Must all the states be Jauch–Piron? Must the quantum logic be infinite? Does it suffice that there are no dispersion-free states? A partial clarification of these questions will be given by the author in a forthcoming paper, in which the results of the present paper will be extended to projection-valued states on von Neumann algebras.

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